Model diagnostics

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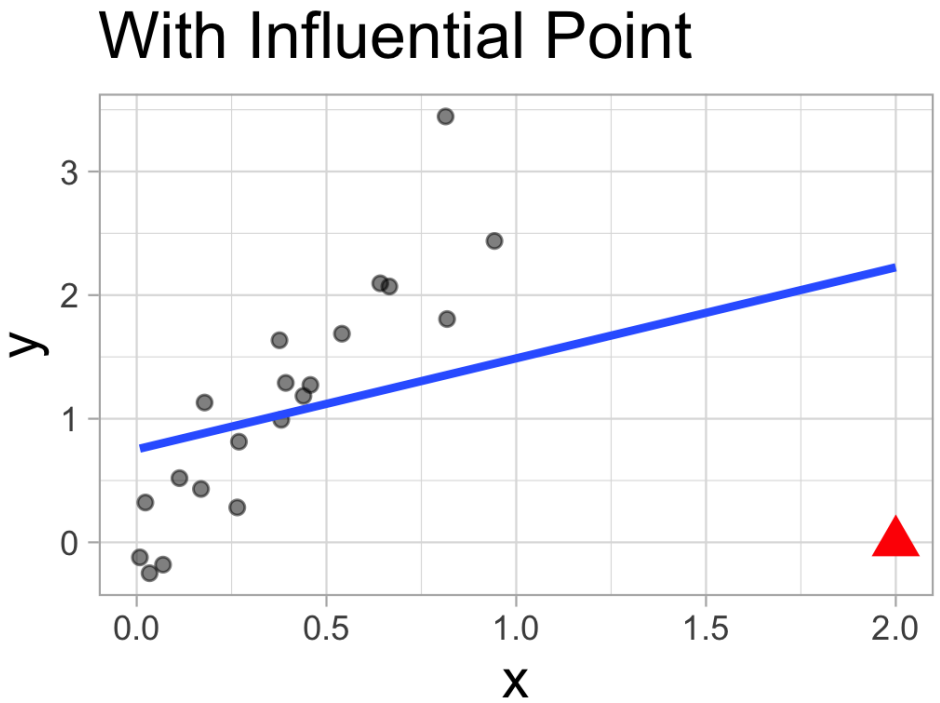
Topics

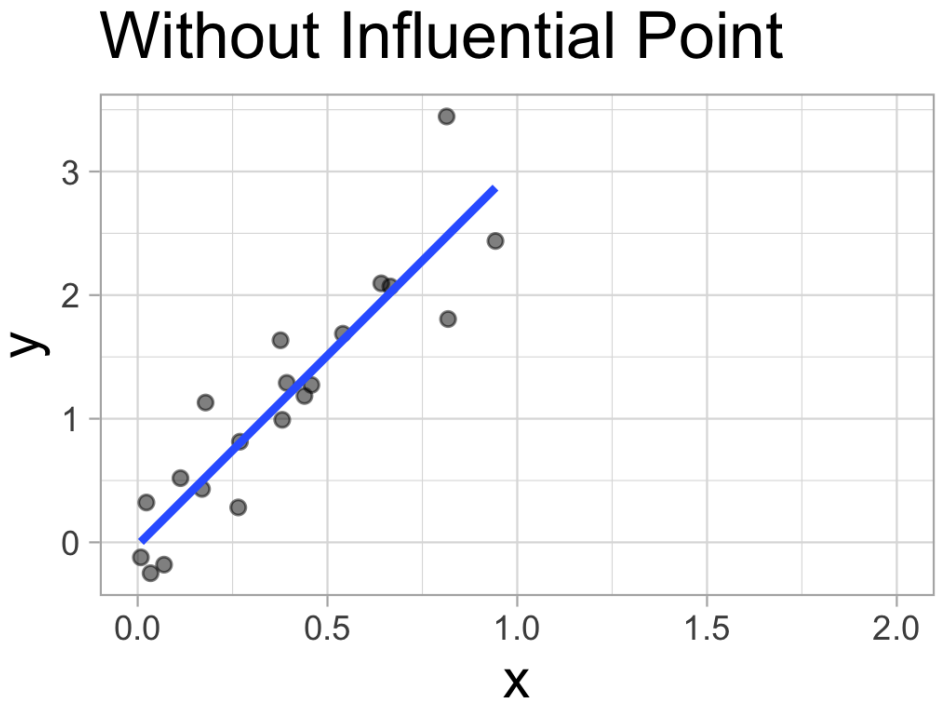
- Identifying influential points
  - Leverage
  - Standardized residuals
  - Cook's Distance
- Multicollinearity

Influential points

Influential Point

An observation is **influential** if removing it substantially changes the coefficients of the regression model





Influential points

- Influential points have a large impact on the coefficients and standard errors used for inference
- These points can sometimes be identified in a scatterplot if there is only one predictor variable
  - This is often not the case when there are multiple predictors
- We will use measures to quantify an individual observation's influence on the regression model
  - **leverage**, **standardized residuals**, and **Cook's distance**

Model diagnostics in R

Use the **`augment`** function in the broom package to output the model diagnostics (along with the predicted values and residuals)

- response and predictor variables in the model
- **`.fitted`**: predicted values
- **`.se.fit`**: standard errors of predicted values
- **`.resid`**: residuals
- **`.hat`**: leverage
- **`.sigma`**: estimate of residual standard deviation when the corresponding observation is dropped from model
- **`.cooksd`**: Cook's distance
- **`.std.resid`**: standardized residuals]

Example: SAT Averages by State

- This data set contains the average SAT score (out of 1600) and other variables that may be associated with SAT performance for each of the 50 U.S. states. The data is based on test takers for the 1982 exam.
- Response variable:
  - **`SAT`**: average total SAT score

Data comes from **`case1201`** data set in the **`Sleuth3`** package

SAT Averages: Predictors

- **`Takers`**: percentage of high school seniors who took exam
- **`Income`**: median income of families of test-takers ($ hundreds)
- **`Years`**: average number of years test-takers had formal education in social sciences, natural sciences, and humanities
- **`Public`**: percentage of test-takers who attended public high schools
- **`Expend`**: total state expenditure on high schools ($ hundreds per student)
- **`Rank`**: median percentile rank of test-takers within their high school classes

Model

term	estimate	std.error	statistic	p.value
(Intercept)	-94.659	211.510	-0.448	0.657
Takers	-0.480	0.694	-0.692	0.493
Income	-0.008	0.152	-0.054	0.957
Years	22.610	6.315	3.581	0.001
Public	-0.464	0.579	-0.802	0.427
Expend	2.212	0.846	2.615	0.012
Rank	8.476	2.108	4.021	0.000

SAT: Augmented Data

```
## Rows: 50
## Columns: 14
## $ SAT        <int> 1088, 1075, 1068, 1045, 1045, 1033, 1028, 1022, 1017, 1011,…
## $ Takers     <int> 3, 2, 3, 5, 5, 8, 7, 4, 5, 10, 5, 4, 9, 8, 7, 3, 6, 16, 19,…
## $ Income     <int> 326, 264, 317, 338, 293, 263, 343, 333, 328, 304, 358, 295,…
## $ Years      <dbl> 16.79, 16.07, 16.57, 16.30, 17.25, 15.91, 17.41, 16.57, 16.…
## $ Public     <dbl> 87.8, 86.2, 88.3, 83.9, 83.6, 93.7, 78.3, 75.2, 97.0, 77.3,…
## $ Expend     <dbl> 25.60, 19.95, 20.62, 27.14, 21.05, 29.48, 24.84, 17.42, 25.…
## $ Rank       <dbl> 89.7, 90.6, 89.8, 86.3, 88.5, 86.4, 83.4, 85.9, 87.5, 84.2,…
## $ .fitted    <dbl> 1057.0438, 1037.6261, 1041.7431, 1021.3039, 1048.4680, 1013…
## $ .resid     <dbl> 30.9562319, 37.3739084, 26.2569334, 23.6961288, -3.4680381,…
## $ .hat       <dbl> 0.11609974, 0.16926150, 0.11000956, 0.06036139, 0.12261873,…
## $ .sigma     <dbl> 26.16716, 25.89402, 26.30760, 26.38760, 26.64972, 26.43025,…
## $ .cooksd    <dbl> 2.931280e-02, 7.051849e-02, 1.970989e-02, 7.901850e-03, 3.9…
## $ .std.resid <dbl> 1.24986670, 1.55651598, 1.05649773, 0.92792786, -0.14054225…
## $ obs_num    <int> 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, …
```

Leverage

Leverage

- **Leverage:** measure of the distance between an observation's values of the predictor variables and the average values of the predictor variables for the entire data set
- An observation has **high leverage** if its combination of values for the predictor variables is very far from the typical combination of values in the data
- Observations with high leverage should be considered as *potential* influential points

Calculating leverage

Simple Regression: leverage of the i^{th} observation

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{j=1}^{n}(x_j - \bar{x})^2}$$

Multiple Regression: leverage of the i^{th} observation is the i^{th} diagonal of

$$\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$$

- *Note*: Leverage only depends on values of the **predictor** variables

High Leverage

The sum of the leverages for all points is $p + 1$

- In the case of SLR $\sum_{i=1}^{n} h_i = 2$
- The "typical" leverage is $\frac{(p+1)}{n}$

An observation has **high leverage** if

$$h_i > \frac{2(p+1)}{n}$$

High Leverage

If there is point with high leverage, ask

❓ Is there a data entry error?

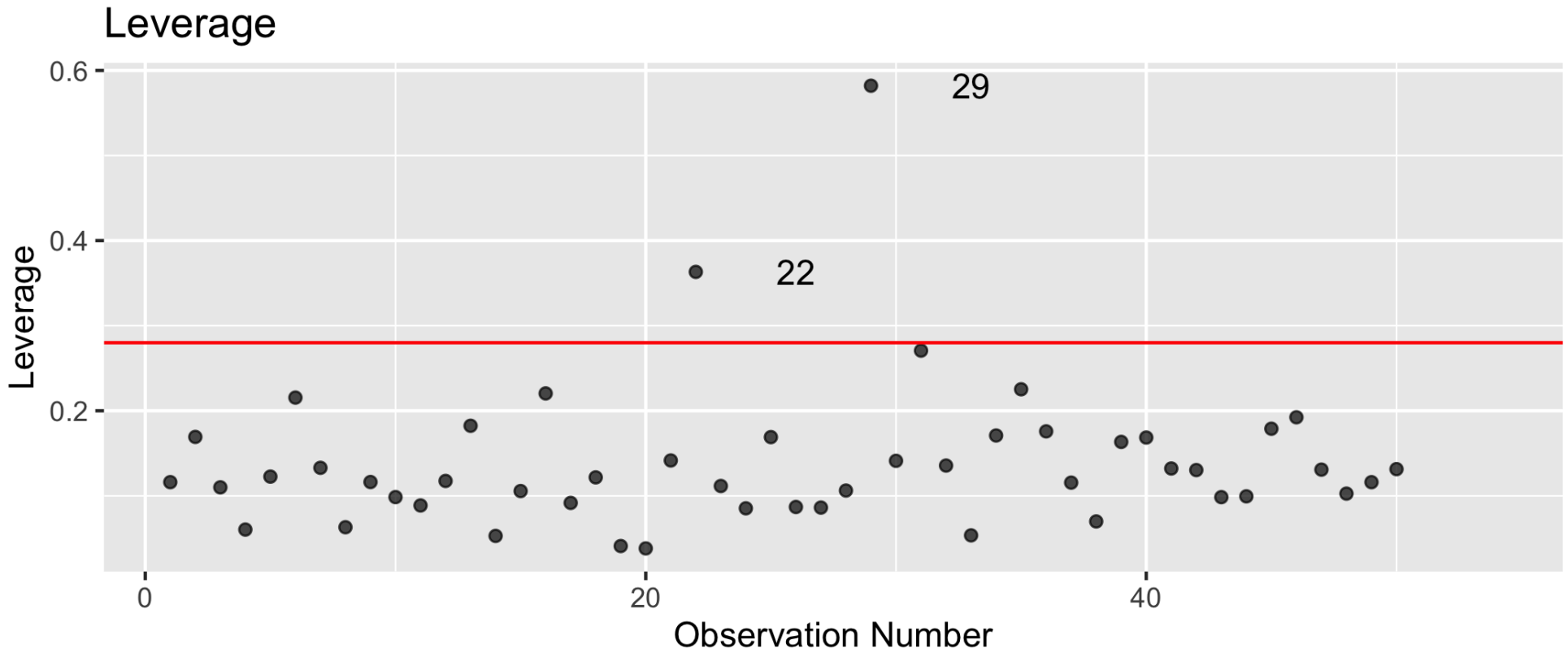
❓ Is this observation within the scope of individuals for which you want to make predictions and draw conclusions?

❓ Is this observation impacting the estimates of the model coefficients, especially for interactions?

Just because a point has high leverage does not necessarily mean it will have a substantial impact on the regression. Therefore we need to check other measures.

SAT: Leverage

High leverage if $h_i > \frac{2*(6+1)}{50} = 0.28$

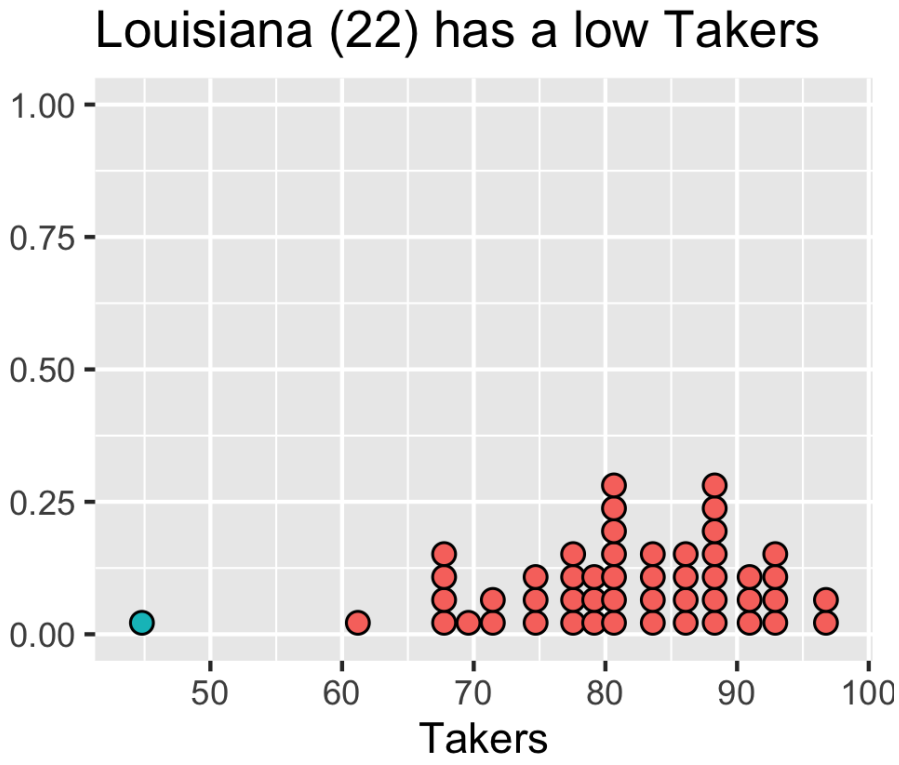


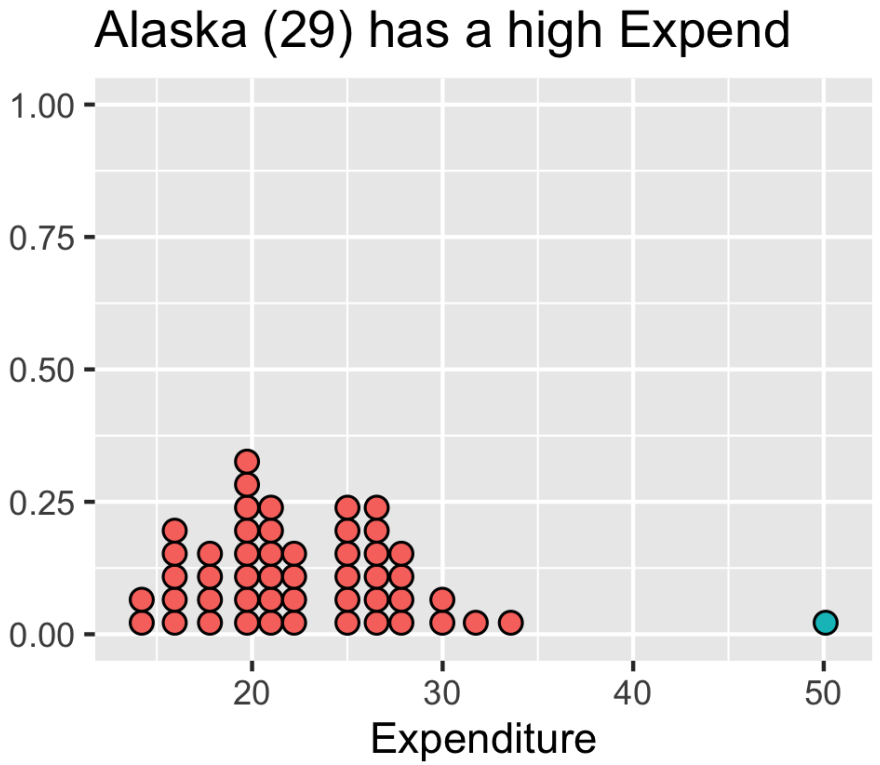
Observations with high leverage

obs_num	Takers	Income	Years	Public	Expend	Rank
22	5	394	16.85	44.8	19.72	82.9
29	31	401	15.32	96.5	50.10	79.6

Why do you think these observations have high leverage?

Let's dig into the data





Standardized & Studentized Residuals

Standardized & Studentized Residuals

- What is the best way to identify outliers (points that don't fit the pattern from the regression line)?
- Look for points that have large residuals
- We want a common scale, so we can more easily identify "large" residuals
- We will look at each residual divided by its standard error

Standardized & Studentized residuals

$$std.res_i = \frac{y_i - \hat{y}_i}{\hat{\sigma}_\epsilon \sqrt{1 - h_i}}$$

where $\hat{\sigma}_\epsilon$ is the regression standard error

$$stud.res_i = \frac{y_i - \hat{y}_i}{\hat{\sigma}_{(i)} \sqrt{1 - h_i}}$$

where $\hat{\sigma}_{(i)}$ is the regression standard error from fitting the model with the i^{th} point removed

Standardized & Studentized residuals

- Observations with high leverage tend to have low values of standardized residuals because they pull the regression line towards them
- This issue is avoided using the studentized residuals, since the regression standard error is calculated without the possible influential point.
- Standardized residuals are produced by **`augment`** in the column **`.std.resid`**
- Studentized residuals can be calculated using **`.sigma`**, **`.resid`**, and **`.hat`** produced by **`augment`**

Using standardized & studentized residuals

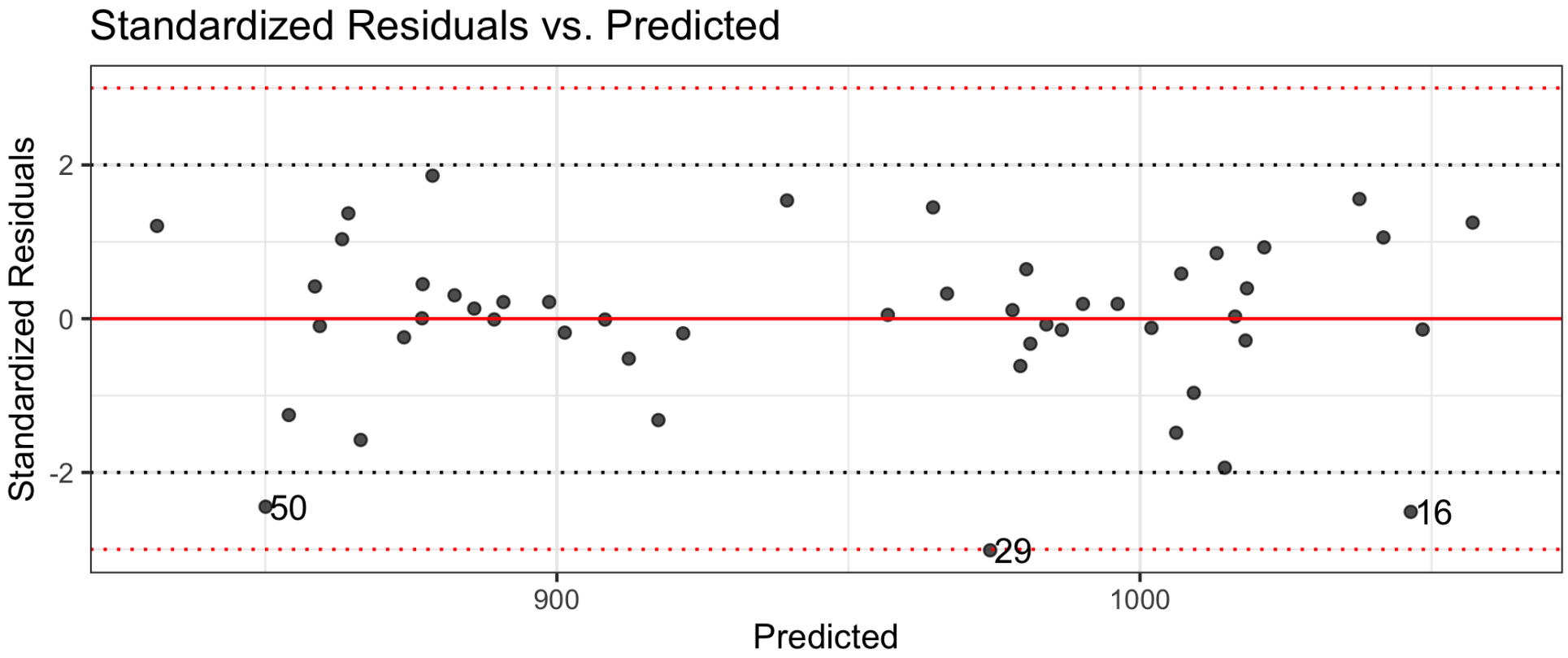
Observations that have standardized residuals of large magnitude are outliers, since they don't fit the pattern determined by the regression model

An observation is a **moderate outlier** if its standardized residual is beyond ± 2.

An observation is a **serious outlier** if its standardized residual is beyond ± 3.

Make residual plots with standardized residuals to make it easier to identify outliers and check constant variance condition.

SAT: Standardized residuals vs. predicted



Cook's Distance

Motivating Cook's Distance

An observation's influence on the regression line depends on

- How close it lies to the general trend of the data - $std.resid_i$
- Its leverage - h_i

Cook's Distance is a statistic that includes both of these components to measure an observation's overall impact on the model

Cook's Distance

Cook's distance for the i^{th} observation

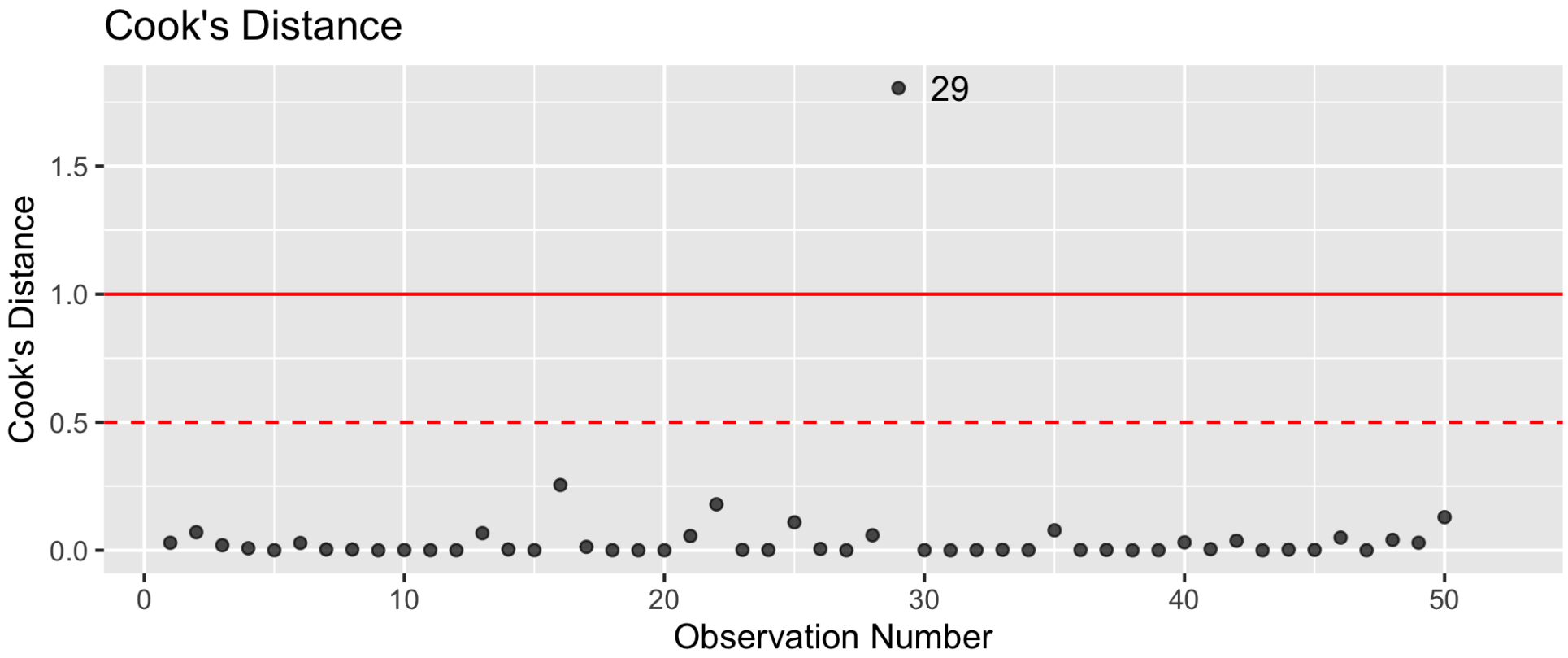
$$D_i = \frac{(std.res_i)^2}{p+1}\left(\frac{h_i}{1-h_i}\right)$$

An observation with large D_i is said to have a strong influence on the predicted values

An observation with

- $D_i > 0.5$ is **moderately influential**
- $D_i > 1$ is **very influential**

Cook's Distance



Influential point: Alaska

```
## # A tibble: 1 x 7
##   obs_num Takers Income Years Public Expend  Rank
##     <int>  <int>  <int> <dbl>  <dbl>  <dbl> <dbl>
## 1      29     31    401  15.3   96.5   50.1  79.6
```

- high leverage 0.5820757
- large magnitude standardized residual -3.0119952

Model with and without Alaska

With Alaska

term	estimate
(Intercept)	-94.659
Takers	-0.480
Income	-0.008
Years	22.610
Public	-0.464
Expend	2.212
Rank	8.476

Without Alaska

term	estimate
(Intercept)	-203.926
Takers	0.018
Income	0.181
Years	16.536
Public	-0.443
Expend	3.730
Rank	9.789

Using these measures

- Standardized residuals, leverage, and Cook's Distance should all be examined together
- Examine plots of the measures to identify observations that are outliers, high leverage, and may potentially impact the model.

What to do with outliers/influential points?

It is **OK** to drop an observation based on the **predictor variables** if...

- It is meaningful to drop the observation given the context of the problem
- You intended to build a model on a smaller range of the predictor variables. Mention this in the write up of the results and be careful to avoid extrapolation when making predictions

What to do with outliers/influential points?

It is **not OK** to drop an observation based on the response variable

- These are legitimate observations and should be in the model
- You can try transformations or increasing the sample size by collecting more data

In either instance, you can try building the model with and without the outliers/influential observations

See the supplemental notes Details on Model Diagnostics for more details about standardized residuals, leverage points, and Cook's distance.

Multicollinearity

Why multicollinearity is a problem

- We can't include two variables that have a perfect linear association with each other
- If we did so, we could not find unique estimates for the model coefficients

Example

Suppose the true population regression equation is $y = 3 + 4x$

- Suppose we try estimating that equation using a model with variables x and $z = x/10$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 z$$

$$= \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 \frac{x}{10}$$

$$= \hat{\beta}_0 + \left(\hat{\beta}_1 + \frac{\hat{\beta}_2}{10}\right) x$$

Example

$$\hat{y} = \hat{\beta}_0 + \left(\hat{\beta}_1 + \frac{\hat{\beta}_2}{10}\right)x$$

- We can set $\hat{\beta}_1$ and $\hat{\beta}_2$ to any two numbers such that $\hat{\beta}_1 + \frac{\hat{\beta}_2}{10} = 4$
- Therefore, we are unable to choose the "best" combination of $\hat{\beta}_1$ and $\hat{\beta}_2$

Why multicollinearity is a problem

- When we have almost perfect collinearities (i.e. highly correlated predictor variables), the standard errors for our regression coefficients inflate
- In other words, we lose precision in our estimates of the regression coefficients
- This impedes our ability to use the model for inference or prediction

Detecting Multicollinearity

Multicollinearity may occur when...

- There are very high correlations ($r > 0.9$) among two or more predictor variables, especially when the sample size is small
- One (or more) predictor variables is an almost perfect linear combination of the others
- Include a quadratic in the model mean-centering the variable first
- Including interactions between two or more continuous variables

Detecting multicollinearity in the EDA

✅ Look at a correlation matrix of the predictor variables, including all indicator variables

- Look out for values close to 1 or -1

✅ Look at a scatterplot matrix of the predictor variables

- Look out for plots that show a relatively linear relationship

Detecting Multicollinearity (VIF)

Variance Inflation Factor (VIF): Measure of multicollinearity in the regression model

$$VIF(\hat{\beta}_j) = \frac{1}{1 - R^2_{X_j|X_{-j}}}$$

where $R^2_{X_j|X_{-j}}$ is the proportion of variation X that is explained by the linear combination of the other explanatory variables in the model.

Detecting Multicollinearity (VIF)

Typically $VIF > 10$ indicates concerning multicollinearity

- Variables with similar values of VIF are typically the ones correlated with each other

Use the `vif()` function in the **rms** R package to calculate VIF

VIF For SAT Model

```
vif(sat_model) %>% tidy() %>% kable()
```

names	x
Takers	16.478636
Income	3.128848
Years	1.379408
Public	2.288398
Expend	1.907995
Rank	13.347395

Takers and **Rank** are correlated. We need to remove one of these variables and refit the model.

Model without `Takers`

term	estimate	std.error	statistic	p.value
(Intercept)	-213.754	122.238	-1.749	0.087
Income	0.043	0.133	0.322	0.749
Years	22.354	6.266	3.567	0.001
Public	-0.559	0.559	-0.999	0.323
Expend	2.094	0.824	2.542	0.015
Rank	9.803	0.872	11.245	0.000

```
## # A tibble: 1 x 3
##   adj.r.squared   AIC   BIC
##           <dbl> <dbl> <dbl>
## 1         0.863  476.  489.
```

Model without Rank

term	estimate	std.error	statistic	p.value
(Intercept)	535.091	164.868	3.246	0.002
Income	-0.117	0.174	-0.675	0.503
Years	26.927	7.216	3.731	0.001
Public	0.536	0.607	0.883	0.382
Expend	2.024	0.980	2.066	0.045
Takers	-3.017	0.335	-9.014	0.000

```
## # A tibble: 1 x 3
##   adj.r.squared   AIC   BIC
##           <dbl> <dbl> <dbl>
## 1         0.814  491.  505.
```

Recap

- Identifying influential points
  - Leverage
  - Standardized residuals
  - Cook's Distance
- Multicollinearity